

THRESHOLDS FOR PLAUSIBLE ANALOGICAL ARGUMENTS

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ABSTRACT

This paper argues that analogical arguments, especially in science, are often employed to show that their conclusions are *prima facie* plausible, or serious possibilities. *Prima facie* plausibility is not a matter of degree; rather, it implies the existence of a threshold below which analogical arguments provide no justification for their conclusions. This position is supported by means of analogical arguments in mathematics. The paper then argues that structure-mapping theories cannot easily accommodate the notion of *prima facie* plausibility.

INTRODUCTION

This paper has two main objectives. The first is to articulate and defend a simple normative principle of analogical reasoning. The principle is easily stated.

Threshold principle. An analogical argument must meet a *threshold*, a non-trivial standard of justification, in order to provide any plausibility for its conclusion.

The second objective is to demonstrate that despite its apparent innocence, this principle is incompatible with leading computational theories of analogy.

I begin by first explaining, and then offering historical and philosophical motivation for, the Threshold principle. I then discuss three examples that reveal important features of analogical reasoning in mathematics, features that provide further

support for the Threshold principle. Next, I explain why the principle appears to be incompatible with computational models of analogical reasoning—specifically, with structure-mapping theories. I conclude by showing that the Threshold principle is also important outside mathematics, and by explaining why it should play a crucial role in understanding analogical reasoning.

1. The Threshold principle and the modal conception of plausibility

The Threshold principle asserts that there is some non-trivial threshold such that analogical arguments falling ‘below the cut’ provide no support for their conclusions. It complements a widely recognized norm.

Degree principle. Analogical arguments vary in the *degree* of plausibility that they provide for their conclusions.

If both principles are accepted, then a good theory of analogical arguments should provide *both* a threshold for acceptability *and*, for arguments that make the grade, some means of assessing their degree of plausibility. I shall argue (in section 3) that structure-mapping theories respect only the second of these requirements.

To appreciate that the Threshold principle is neither trivial nor an entirely new idea, we turn to ideas about analogy expressed by 19th century philosophers of science. John Stuart Mill, in his analysis of analogical arguments in *A System of Logic*, writes:

There can be no doubt that every resemblance [not known to be irrelevant] affords some degree of probability, beyond what would otherwise exist, in favour of the conclusion (1843/1930, 333).

This is the *Degree* principle without the *Threshold* principle. Every resemblance counts for something; there are better and worse analogical arguments, but no cut-off point. On this view, any proposed scientific hypothesis supported by an analogy has a measure of support.

By contrast, John Herschel and William Whewell insist that for a proposed hypothesis to merit serious attention, it must postulate a *vera causa* (true cause). As Snyder explains, for Herschel this means that the hypothesis must be “analogous to causes that are already known to have produced similar effects in other cases” (Snyder 2006, 201)—that is, to causes known to exist in nature. What is more, frivolous analogies will not do. Herschel appears to be defending the idea of a minimum standard that we use to screen proposed hypotheses before we even get to the stage of serious testing.

Mill, Herschel and Whewell all agree that analogical arguments in science can be used to show that a hypothesis is *plausible*.¹ This means, roughly, that we have some reason to believe it. But our brief discussion already reveals an important ambiguity as to whether an assertion of plausibility is categorical or a matter of degree. This ambiguity can be cleared up by distinguishing between three separate notions of plausibility.

On the *probabilistic* conception, plausibility is identified with subjective credence, and is a matter of *degree* of belief. To assert that a hypothesis is highly plausible, for example, is to say that it has high

probability. If a new drug proves highly successful in treating lab animals for heart disease, we might infer by analogy that it is highly plausible that the drug will help humans.

On the *comparative* conception, plausibility judgments involve a contrast between two hypotheses: one is more plausible than the other, or else they are equally plausible. Like the probabilistic conception, the comparative view expresses a version of the idea that plausibility comes in degrees.

Finally, I suggest that there is also a *modal* conception, according to which plausibility is an attribute of a single hypothesis, but not a matter of degree. To assert that a hypothesis is plausible in the modal sense is just to assert that we have sufficient reason to take it seriously, i.e., to investigate it further. It is to single out the hypothesis from an undifferentiated mass of bare possibilities. Here I echo ideas expressed by the physicist Norman Campbell, writing about the role of analogy in Fourier’s theory of heat conduction:

Some analogy is essential to it; for it is only this analogy which distinguishes the theory from the multitude of others . . . which might also be proposed to explain the same laws (1957, 142).

A good analogy with a known theory justifies us in regarding Fourier’s theory as a serious possibility, even prior to testing. This type of plausibility judgment is not comparative and it is not a matter of degree. Rather, it is the application of a modal construct—“It is a serious possibility that”—to a hypothesis. I call this a judgment of *prima facie* plausibility. I don’t believe that this conception reduces to the probabilistic one. I will return to this point at the end of section 3.

Analogical arguments may be directed towards establishing any of the three kinds of plausibility. My contention, however, is that

¹ This view about analogies is widely shared, but not universal (see (Morrison 2000)).

the modal conception is fundamental in at least three ways.

- **For individual analogical arguments.** The comparative notion is inapplicable in situations where we want to evaluate an *individual analogical argument*, one that involves a single source domain, a single target domain, and a single hypothesis. Furthermore, the probabilistic notion is inapplicable if our objective is merely to show that the hypothesis is a serious possibility.
- **For hypothesis screening.** An analogical argument that a hypothesis is *prima facie* plausible is often all that we want. In particular, such an argument is adequate for the screening process that goes on in the early stages of a scientific inquiry, when we simply want to identify hypotheses worthy of serious consideration.
- **As a preliminary step in all analogical reasoning.** There is no point in applying either the comparative or probabilistic notion of plausibility to a hypothesis unless it is a serious possibility.

I shall focus on the first point: individual analogical arguments are often directed towards establishing *prima facie* plausibility. Do we ever evaluate analogical arguments in isolation? Do we evaluate them in a non-probabilistic, non-comparative fashion? A consideration of how analogies work in mathematics provides positive answers to these questions.

2. Mathematical analogies

Analogical arguments are important in mathematics, and analogical arguments in mathematics are important for anybody who wants to understand analogical reasoning. The use of analogy in mathematics does not *seem*,

on the face of it, to be all that different from its use in other domains. I'll present three examples and then explain how they support the claims of the previous section. In section 4, I'll argue that the conclusion generalizes to non-mathematical analogies.

Example 1 (*triangles and tetrahedra*). Suppose we have proved that the three medians of any triangle have a common intersection point. By analogy, we conjecture that the four medians of any tetrahedron—the lines joining each vertex with the center of the opposite face—have a common intersection. In this geometrical analogy, the source domain is plane geometry, and the feature being transferred to the target is that the medians are concurrent—that is, they intersect in a common point.

Example 2 (*rectangles and boxes*). Our source domain consists of rectangles in the plane, with sides $x, y > 0$. The target domain consists of rectangular boxes with sides $x, y, z > 0$. Let Q be the theorem that of all rectangles with a fixed perimeter ($x + y = \text{constant}$), the square has maximum area. Let Q^* be the analogous conjecture that of all rectangular boxes with a fixed perimeter ($x + y + z = \text{constant}$), the largest such box is a cube. We argue by analogy that Q^* is plausible.

Example 3 (*complex numbers and operators*). The source domain is the set of complex numbers, and the target is the set of linear operators on V , where V is a finite-dimensional inner product space. Operations with complex numbers (addition, multiplication, and conjugation) are similar to operations involving linear operators (addition, function composition, and the adjoint map). Part of the analogy is represented in the following table.

Thresholds for Plausible Analogical Arguments

Complex Numbers

$$\overline{z + w} = \overline{z} + \overline{w}$$

=

$$z = z$$

$$\overline{zw} = \overline{w} \cdot \overline{z}$$

Linear Operators

$$(B+A)^* = B^* + A^*$$

$$A^{**} = A$$

$$(BA)^* = A^*B^*$$

One important and well-known disanalogy is that complex multiplication is commutative ($zw = wz$), but operator composition is not commutative (in general, $BA \neq AB$).

Consider the set of *symmetric* operators, consisting of those A such that $A^* = A$. This class is analogous to the subset of real numbers—that is, the set of complex numbers such that $\overline{z} = z$. We know that the sum and product of two real numbers are real. We might conjecture, by analogy, that the sum and product of two symmetric operators are symmetric.

None of these examples is especially inventive. All three involve the mundane application of analogical reasoning to solve problems. Still, the examples share some interesting features.

1) *Individual analogical argument.* In each case, we have a single source and a single target domain. No comparison to any rival conjecture, explicit or implied, is needed to make sense of the argument.

2) *Modal conclusion.* In each case, the analogy is alleged to support a conclusion that is best captured in modal terms: the conjecture is *prima facie* plausible and merits further investigation. To insist that we put this

conclusion in probabilistic terms would be a distraction.²

3) *Used for hypothesis screening.* In each case, we can distinguish between the *psychological* role of the analogy in helping us to generate a hypothesis (or conjecture), and the *logical* role of the analogical argument, which is to justify treating the conjecture as a serious possibility.

4) *Non-trivial plausibility criteria.* In each example, we can make a substantive assessment of whether or not the analogical argument succeeds in establishing its conclusion as *prima facie* plausible.

Hoping that the first three items are now sufficiently clear so as not to require further argument, I concentrate on the last point. The claim that there is a threshold for *prima facie* plausibility is uninteresting if the threshold is so low as to be trivial. When we consider the three examples, we certainly find similarities—deep structural similarities—between the source and target domains. On Mill's theory, or on structure-mapping theories, that's sufficient for each argument to confer some *degree* of plausibility on its conclusion. If that's all it takes for a hypothesis to be *prima facie* plausible, then the modal conception of plausibility doesn't deserve much attention.

But Mill's theory is wrong, at least for mathematical analogies. A consideration of our three examples shows that there is a substantive but straightforward means of evaluating whether the analogical arguments establish their conclusions as *prima facie* plausible. The basic idea is to consider not just overall similarity (even structural similarity) between domains, but rather

² For example, we would then have to justify the application of subjective probabilities to mathematical propositions.

whether or not the mathematical proof employed in the source domain is “fit for imitation” (Polya 1954, 46) in the target domain. To appreciate this point, let’s briefly review each example in turn.

In the first example, we might start by using analytic geometry to prove the result about medians in the source domain. With vertices A, B, C, the three mid-points are

$$X = \frac{A+B}{2}, Y = \frac{B+C}{2}, \text{ and } Z = \frac{A+C}{2}.$$

The medians are sets of points

$$\{(1-t)C + tX : 0 \leq t \leq 1\},$$

and similarly for Y and Z. The point

$$\frac{A+B+C}{3}$$

lies on each median, as can be seen by taking $t = \frac{2}{3}$. This proves that the medians are concurrent. Now this proof is “fit for imitation” in the target domain because there are obvious analogs for each step in three dimensions. So we have a plausible analogical argument.

For example 2, the proof begins with the assumption that perimeter is constant—that is, $x + y = c$ for some constant c . Then area xy is maximized when the product $x(c - x)$ is maximized. The First Derivative Test tells us that this occurs when $x = c/2$. Then $y = c/2$ as well, and we have proved that the rectangle with maximum area is a square. Once again, the proof is fit for imitation because there are obvious analogs of perimeter and area, and a well-known analog for the First Derivative Test that we can apply in the target domain. We again have a plausible argument.

With example 3, things go differently. The proof that the sum of two real numbers is real is simple: if $\bar{z} = z$ and $\bar{w} = w$, then

$$\overline{z + w} = \bar{z} + \bar{w} = z + w.$$

There is no obvious block to adapting this proof to apply to operators, so the analogy makes it *prima facie* plausible that the sum of two symmetric operators is symmetric. By contrast, the proof that the product of two real numbers is real uses commutativity of multiplication:

$$\overline{zw} = \overline{w \cdot z} = \overline{wz} = \bar{z}\bar{w}.$$

Since commutativity does not generally hold for operators, the proof is not fit for imitation. The argument is blocked by a *known disanalogy*. The argument fails to establish *prima facie* plausibility for the conjecture that BA is symmetric whenever B and A are symmetric.³

For analogical arguments in mathematics, *prima facie* plausibility should always be evaluated relative to a definite proof. The basic criterion is that the proof be fit for imitation. Crudely, this means that there must be no critical disanalogy that would prevent us from adapting the proof to the target domain. If some feature plays a critical role in the source domain proof, and that feature is known to have no analog in the target, then the analogical argument fails to establish any plausibility for its conclusion, regardless of other similarities between the two domains. In mathematics, at least, the modal conception of plausibility is non-trivial. Furthermore, as in the three examples, the modal conception is frequently all that we need from our analogical arguments. We care about these analogies only insofar as they provide sufficient reason to investigate an idea; we don’t use them to make comparative judgments, or to assign a degree of plausibility.

³The conjecture is indeed false, but a modified version that takes commutativity into account is true.

3. A challenge for structure-mapping theories

How does the evaluation of *prima facie* plausibility differ from what we find in structure-mapping theories? Structure-mapping theories are oriented towards maximizing the overall degree of similarity between source and target domains. So they count *all* structural similarities between higher-order relations as contributing to the strength of the argument. On this view, as for Mill's account, there can be no threshold. On the view I am suggesting, a *focal subset* of critical similarities and differences determines *prima facie* plausibility. Let's consider this point in more detail.

As a preliminary matter, I want to acknowledge that computational theories of analogy are meant to model the *psychological process*, rather than the *logic*, of analogical reasoning. Philosophers and psychologists don't have identical objectives for a theory of analogy. Still, insofar as actual human reasoning incorporates normative elements, a psychologically realistic computational theory ought to do the same. If the Threshold principle is important, then it should be accounted for within a good computational model.

In fact, structure-mapping approaches certainly incorporate at least two basic normative principles:

(MP) *Mapping Principle*. Good analogical reasoning consists in finding an *optimal mapping*.

(SP) *Systematicity Principle*. The goodness of an analogical mapping is largely determined by its *systematicity*, the extent to which it approximates isomorphism (or correspondence between higher-order relations in two domains).

In the introduction to *The Analogical Mind* (Gentner, Holyoak and Kokinov 2001), we

find the following description of these principles:

In her structure-mapping theory, Gentner sets forth the view that analogy entails finding a structural alignment, or mapping, between domains. This alignment between two representational structures is characterized by structural parallelism (consistent, one-to-one correspondences between mapped elements) and systematicity—an implicit preference for deep, interconnected systems of relations governed by higher-order relations, such as causal, mathematical, or functional relations. (8)

The authors go on to describe an “emerging consensus” which sees the mapping process as the core of analogical reasoning. Further, although the descendants of Gentner's original (1983) theory now incorporate additional constraints on the mapping process, the key idea remains the identification of “systematic correspondences”. The more coherent and systematic the mapping is, the better the analogical reasoning. The addition of constraints does not change the basic character of the structure-mapping approach.

Structure-mapping can accommodate the *comparative* concept of plausibility. Given two rival analogical arguments, systematicity provides a reasonable basis for a ranking. Structure-mapping can also accommodate the *probabilistic* concept of plausibility. The degree of plausibility of an analogical argument may be correlated with the degree of systematicity that attaches to the associated mapping. My concern is that there is no good way to account for the *modal* conception of plausibility in terms of MP and SP, the two norms of structure-mapping theories. Systematicity is naturally linked only to the probabilistic and comparative conceptions of plausibility.

Let's consider four strategies that might let us reconcile *prima facie* plausibility with

the structure-mapping approach. First, we might say that only conclusions licensed by the *best* (most systematic) analogy mapping count as *prima facie* plausible. This proposal fails because such a ‘winner-take-all’ approach is far too strict to be true to life. In settings such as the law, we recognize that two or more rival analogical arguments, with incompatible conclusions, might all meet minimum standards for *prima facie* plausibility.

Second, we might go to the opposite extreme and say that any analogy mapping that exhibits *some* systematic correspondence among higher-order relations in the source and target domains establishes the *prima facie* plausibility of its conclusion. This proposal fails because it is too liberal. It violates our intuitions about cases such as *Example 3*, where we find many systematic correspondences but still insist that the argument fails to show that the conjecture about symmetric operators has any plausibility.

A third and more interesting strategy is to challenge the significance of the distinction I made, in section 1, between the ‘threshold’ and ‘degree-of-belief’ conceptions of plausibility. Perhaps only the probabilistic conception is fundamental, and the threshold can be represented as a reasonable cut-off probability value. If the structure-mapping approach can deliver a degree of belief, then it can also deliver a verdict as to whether that degree of belief exceeds the cut-off value. This objection is likely to appeal to adherents of probabilistic (Bayesian) models of belief revision.

In response, it must be admitted that there has to be a relationship between the modal and probabilistic conceptions of plausibility. In particular, any *prima facie* plausible hypothesis will be assigned *some* positive probability value. I do not believe, however, that this relationship can be properly handled by stipulating a particular cut-off value. First, there is no non-arbitrary and general method to obtain such a cut-off. Second, as argued in

section 2, the actual probability value is something of a distraction. A hypothesis is modelled as having a certain probability value *because* it is deemed to be *prima facie* plausible, not the other way around. Finally, a cut-off probability value, no matter how low, would not correctly reflect the point that analogical arguments failing to meet a threshold are utterly rejected. Indeed, a thoroughly probabilistic approach to belief cannot accommodate cut-off values for rejection or acceptance, other than 0 or 1.⁴

The fourth and final strategy I shall consider is to argue that the structure-mapping approach *already incorporates* the essential elements of the threshold approach.⁵ The idea is that if there are known disanalogies between source and target domains that would block an analogical inference, then this prior knowledge of the target already blocks the analogical inference by blocking some intermediate step in the inference. So there is no need for a separate Threshold principle.

To illustrate, consider *Example 3*. In that example, the difficulty is that commutativity of operator multiplication is known to fail in the target domain. In such a case, the inference to commutativity of multiplication in the target is trivially blocked, and therefore structure-mapping also blocks the inference that the product of symmetric operators is symmetric.

My response to this objection is as follows. It may be that, in cases such as *Example 3*, there is *some* way to represent domain knowledge so that the *intermediate step* in the analogical inference (commutativity of operator multiplication) is blocked. But there is still no way for the structure-mapping theory to block the analogical inference to the ultimate conclusion (that the product of

⁴ Even degrees of belief of 0 or 1 can be revised if one has a sufficiently flexible view of conditional probability (McGee 1994).

⁵ This objection was made by an anonymous reviewer.

symmetric operators is symmetric), short of altering the problem by introducing a rival conjecture and turning the reasoning into an assessment of comparative plausibility.

4. Analogical arguments outside mathematics

The features of analogical arguments in mathematics that support the Threshold principle are found in other sciences as well. Individual, non-probabilistic analogical arguments are employed in hypothesis screening, i.e., to show that a proposed hypothesis should be taken seriously.

Darwin appears to have this notion of plausibility in mind in connection with his famous analogy between artificial and natural selection. In a May 1860 letter to Henslow, Darwin reflects upon the logical role of the analogy:

Why may I not invent the hypothesis of Natural Selection (which from the analogy of domestic productions, and from what we know of the struggle of existence and of the variability of organic beings, is, in some very slight degree, in itself probable) and try whether this hypothesis of Natural Selection does not explain (as I think it does) a large number of facts? (1903, letter 100)

Despite the reference to a “slight degree” of probability, Darwin’s objective is not to attach some probability value to his hypothesis. Rather, he wants to establish, by means of the analogy, that his hypothesis meets a minimum standard to be taken seriously and subjected to further study.

I hope to have made a convincing case for the importance of the modal notion of plausibility in understanding particular examples of analogical reasoning. But I don’t want to leave you with the impression that such examples are idiosyncratic uses of analogy, and that *prima facie* plausibility is of marginal

interest. I believe that paying heed to the modal conception of plausibility gives us considerable insight into analogical reasoning in general. It brings out the logical mechanisms that we use to adjudicate between good and bad analogical arguments,

In mathematics, I suggest that *prima facie* plausibility is always assessed relative to a particular proof in the source domain. An acceptable analogical argument requires that we establish that the proof is fit for imitation in the target domain. The generalization of this idea to other settings is that to determine whether an analogical argument establishes the *prima facie* plausibility of its conclusion, we need to identify a focal set of logical, causal and explanatory relations in the source domain. The identification of that focal set of relations, which I call the *prior association* (Bartha 2009), must be part of the analogical argument.

5. Conclusion

Given the successes of the structure-mapping approach to analogy, it is tempting to dismiss what I have called the modal notion of plausibility, and the associated Threshold principle. I believe that would be a mistake. In the first place, as we have seen, many examples of analogical arguments are best understood as directed towards showing that some conclusion is *prima facie* plausible, rather than towards making a comparative or probabilistic assessment. My more fundamental point, however, is that by investigating the threshold for plausibility, the minimal criteria for an analogical argument to be taken seriously, we can uncover the mechanisms that contribute to successful arguments. Structure-mapping succeeds as well as it does because, to a first approximation, overall relational similarity tends to track the presence of these underlying mechanisms. For a more refined *normative* theory of analogy, however, we need to look at the details.

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